Cluster Algebras and their applications to Index Theorem.

Andrey Glubokov[1] Igor Nikolaev[2]

[1]Ave Maria University, FL, USA
[2]St. John's University, NY, USA

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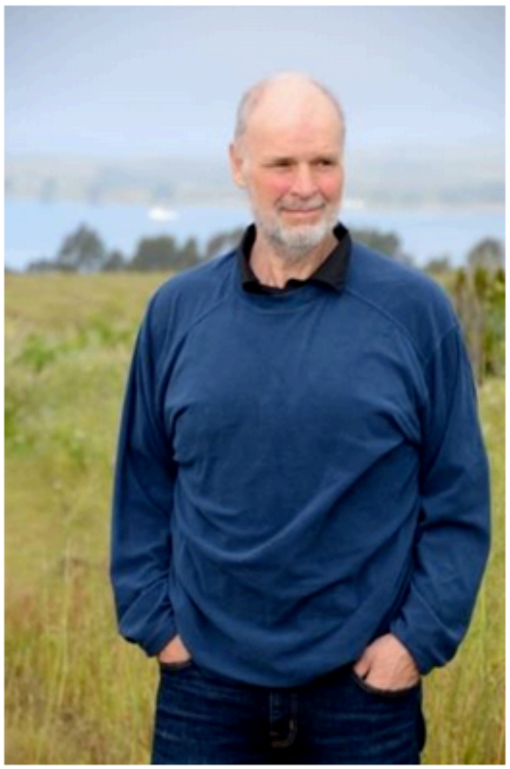

This talk is dedicated to the memory of Sir Vaughan Jones, whose recent untimely death shocked all the Mathematical community.

Sir Vaughan, an eminent scholar joined Vanderbilt University in 2011 as a Distinguished Professor of mathematics. He was also Professor Emeritus at University of California, Berkeley and Distinguished Alumni Professor at the University of Auckland.

Sir Vaughan was a recipient of the Fields Medal, which is widely regarded as the "Nobel Prize of Mathematics," in 1990 and famously wore the New Zealand Rugby jersey when he gave his acceptance speech in Kyoto.

The recognition was, in part, because of his discovery of a relationship between Von Neumann algebras and geometric topology. The discovery led to his finding a new polynomial invariant for knots and links in three-dimensional space – something that had been missed by topologists during the preceding 60 years.

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Cluster Algebras

Definitions.

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The matrix is sometimes assumed to be skew-symmetric, so that $b_{x,y} = -b_{y,x}$ for all x and y. More generally the matrix might be skew-symmetrizable, meaning there are positive integers d_x associated with the elements of the cluster such that $d_x b_{x,y} = -d_y b_{y,x}$ for all x and y.

We denote by A(b, c) a cluster algebra of rank 2 which has exchange matrices

$$\pm \begin{bmatrix} 0 & b \\ -c & 0 \end{bmatrix}$$

Cluster algebras of rank 2

Suppose that we start with the cluster $\{x_1, x_2\}$ and take the exchange matrix with $b_{12} = -b_{21} = 1$.

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The variables are related by

$$x_{n-1}x_{n+1} = 1 + x_n$$

$$x_1\,,\ x_2\,,\quad x_3 = \frac{1 + x_2}{x_1}\,,$$

$$x_1\,,\ x_2\,,\quad x_3 = \frac{1+x_2}{x_1}\,,\qquad x_4 = \frac{1+x_3}{x_2} = \frac{1+x_1+x_2}{x_1 x_2}\,,$$

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$$x_5 = \frac{1+x_4}{x_3} = \frac{1+x_1}{x_2}\,,\ x_6 = \frac{1+x_5}{x_4} = x_1\,,\ x_7 = \frac{1+x_6}{x_5} = x_2\,,\ \ldots$$

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There are similar examples with $b_{12} = 1$, $-b_{21} = 2$ or 3, where the analogous sequence of cluster variables repeats with period 6 or 8. These are also of finite type, and are associated with the Dynkin diagrams B_2 and G_2.

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However if $|b_{12}b_{21}| \geq 4$ then the sequence of cluster variables is not periodic and the cluster algebra is of infinite type.

The cluster variables in the cluster algebra A(1, 2) satisfy the recurrence

$$z_{k-1}\, z_{k+1} = \begin{cases} z_k^2 + 1 & \text{if } k \text{ is even;} \\ z_k + 1 & \text{if } k \text{ is odd.} \end{cases}$$

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Expressing everything in terms of the initial cluster (z_1, z_2), we get:

$$z_3 = \frac{z_2^2 + 1}{z_1}, \quad z_4 = \frac{z_2^2 + z_1 + 1}{z_1 z_2}, \quad z_5 = \frac{z_1^2 + z_2^2 + 2z_1 + 1}{z_1 z_2^2}, \quad z_6 = \frac{z_1 + 1}{z_2},$$

and then $z_7 = z_1$ and $z_8 = z_2$, so the sequence is 6-periodic! Thus in this case, we have only 6 distinct cluster variables.

What is Jones Index Theorem?

The Jones Index Theorem is an analog of the Galois theory for the von Neumann algebras [V.F.R. Jones, *Subfactors and Knots*, CBMS Series **80**, AMS, 1991] [5].

The **von Neumann algebra** is a *-algebra of bounded operators on a Hilbert space that is closed in the weak operator topology and contains the identity operator.

The **factor** is a von Neumann algebra $\mathscr{M}$ with the trivial center.

A **subfactor** $\mathscr{N}$ of the factor $\mathscr{M}$ is a subalgebra, such that $\mathscr{N}$ is a factor.

A factor is said to be of **type II** if there are no minimal projections but there are non-zero finite projections.

The **index**

$$[\mathscr{M} : \mathscr{N}]$$

of a subfactor $\mathscr{N}$ of a type II factor $\mathscr{M}$ is a positive real number $\dim_{\mathscr{N}}(L^2(\mathscr{M}))$, where $L^2(\mathscr{M})$ is a representation of $\mathscr{N}$ obtained from from the canonical trace on $\mathscr{M}$ using the Gelfand-Naimark-Segal construction.

Jones basic construction

denote by e_{ij} the matrix units of the algebra $M_2(\mathbf{C})$.
Then

$$e_t = \frac{1}{1+t}(e_{11} \otimes e_{11} + te_{22} \otimes e_{22} + \sqrt{t}(e_{12} \otimes e_{21} + e_{21} \otimes e_{12}))$$

is a projection of the algebra $M_2(\mathbf{C}) \otimes M_2(\mathbf{C})$ for each $t \in \mathbf{R}$.

Proceeding by induction, one can define projections $e_i(t) = \theta^i(e_t) \in M_{2^i}$, where θ is the shift automorphism of the Uniformly hyperfinite algebra (*UHF*-algebra) [1] M_{2^∞}.

[1] can be written as the closure, in the norm topology, of an increasing union of finite-dimensional full matrix algebras.

The $e_i := e_i(t)$ satisfy the following relations

$$\left\{ \begin{array}{rcl} e_i e_j & = & e_j e_i, \quad \text{if} \quad |i-j| \geq 2 \\ e_i e_{i\pm 1} e_i & = & \frac{t}{(1+t)^2} e_i. \end{array} \right. \tag{1}$$

Recall that projections $e_i(t)$ are critical for construction of a subfactor $\mathscr{N}$ of the type II von Neumann algebra $\mathscr{M}$, such that

$$[\mathscr{M} : \mathscr{N}]^{-1} = \frac{t}{(1+t)^2}, \qquad (2)$$

The **Jones Index Theorem** says that

$$[\mathscr{M} : \mathscr{N}] \in [4, \infty) \bigcup \{4\cos^2\left(\frac{\pi}{n}\right) \mid n \geq 3\}. \tag{3}$$

Cluster Algebra

Cluster algebras were introduced by S.Fomin and A.Zelevinsky, *Cluster algebras I: Foundations*, J. Amer. Math. Soc. **15** (2002), 497-529. [3].

Definition

A **cluster algebra** $\mathscr{A}(\mathbf{x}, B)$ of rank n is a subring of the field of rational functions in n variables depending on a cluster of variables $\mathbf{x} = (x_1, \ldots, x_n)$ and a skew-symmetric matrix $B = (b_{ij}) \in M_n(\mathbf{Z})$; the pair $(\mathbf{x}, B)$ is called a seed.

A new cluster $\mathbf{x}' = (x_1, \ldots, x_k', \ldots, x_n)$ and a new skew-symmetric matrix $B' = (b_{ij}')$ is obtained from $(\mathbf{x}, B)$ by the exchange relations:

$$\begin{aligned} x_k x_k' &= \prod_{i=1}^{n} x_i^{\max(b_{ik},0)} + \prod_{i=1}^{n} x_i^{\max(-b_{ik},0)}, \\ b_{ij}' &= \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k \\ b_{ij} + \frac{|b_{ik}|b_{kj} + b_{ik}|b_{kj}|}{2} & \text{otherwise.} \end{cases} \end{aligned} \qquad (4)$$

The seed $(\mathbf{x}', B')$ is said to be a mutation of $(\mathbf{x}, B)$ in direction k, where $1 \leq k \leq n$; the algebra $\mathscr{A}(\mathbf{x}, B)$ is generated by cluster variables $\{x_i\}_{i=1}^{\infty}$ obtained from the initial seed $(\mathbf{x}, B)$ by the iteration of mutations in all possible directions k.

The Laurent phenomenon says that $\mathscr{A}(\mathbf{x}, B) \subset \mathbf{Z}[\mathbf{x}^{\pm 1}]$, where $\mathbf{Z}[\mathbf{x}^{\pm 1}]$ is the ring of the Laurent polynomials in variables $\mathbf{x} = (x_1, \ldots, x_n)$ depending on an initial seed $(\mathbf{x}, B)$.

The $\mathscr{A}(\mathbf{x}, B)$ is a commutative algebra with an additive abelian semigroup consisting of the Laurent polynomials with positive coefficients. Thus the algebra $\mathscr{A}(\mathbf{x}, B)$ is a countable abelian group with an order satisfying the Riesz interpolation property, i.e. a dimension group [Effros 1981, Theorem 3.1]

A *cluster C^*-algebra* $\mathbb{A}(\mathbf{x}, B)$ is an Approximately finite-dimentional (AF) -algebra, such that $K_0(\mathbb{A}(\mathbf{x}, B)) \cong \mathscr{A}(\mathbf{x}, B)$, where $\cong$ is an isomorphism of the dimension groups [2].

[2] I. Nikolaev, *On cluster C^*-algebras*, J. Funct. Spaces **2016**, Art. ID 9639875, 8 pp.

Cluster Algebras of rank 2

Let us consider the field of rational functions in two commuting independent variables x_1 and x_2 with the rational coefficients. For a pair of positive integers b and c, we define elements x_i by the exchange relations

$$x_{i-1}x_{i+1} = \begin{cases} 1 + x_i^b & \text{if} \quad i \text{ odd,} \\ 1 + x_i^c & \text{if} \quad i \text{ even.} \end{cases} \tag{5}$$

By a *cluster algebra rank 2* we understand the algebra $\mathscr{A}(b,c)$ generated by the cluster variables x_i [3]

Denote by $\mathcal{B}$ a basis of the algebra $\mathscr{A}(b,c)$.

Theorem

[Sherman-Zelevinsky]

Suppose that $b = c = 2$ *or* $b = 1$ *and* $c = 4$*. Then* $\mathcal{B} = \{x_i^p x_{i+1}^q \mid p, q \geq 0\} \bigcup \{T_n(x_1x_4 - x_2x_3) \mid n \geq 1\}$*, where* $T_n(x)$ *are the Chebyshev polynomials of the first kind.*

[3] P. Sherman and A. Zelevinsky, *Positivity and canonical bases in rank 2 cluster algebras of finite and affine types*, Moscow Math. J. **4** (2004), 947-974.

Let $r < R$ and consider an annulus in the complex plane

$$\mathscr{D} = \{z = x + iy \in \mathbf{C} \mid r \leq |z| \leq R\}. \tag{6}$$

The Riemann surfaces $\mathscr{D}$ and $\mathscr{D}'$ are isomorphic (conformally equivalent) if and only if $t := R/r = R'/r'$. By

$$T_{\mathscr{D}} = \{t \in \mathbf{R} \mid t > 1\} \tag{7}$$

we understand the Teichmüller space of $\mathscr{D}$. It is known that the cluster algebra $\mathscr{A}(\mathbf{x}, B)$ given by a matrix

$$B = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} \tag{8}$$

encodes the Penner coordinates on $T_{\mathscr{D}}$ [Fomin, Shapiro & Thurston 2008] [4, Example 4.4] and [Williams 2014] [8, Section 3]. We denote by $\mathbb{A}(\mathscr{D}) := \mathbb{A}(\mathbf{x}, B)$ a cluster C^*-algebra corresponding to the matrix B.

Consider the Uniformly hyperfinite (UHF)-algebra of the form $M_{2^\infty} := \bigotimes_{i=1}^{\infty} M_2(\mathbf{C})$. There exits an embedding [4]

$$\mathbb{A}(\mathscr{D}) \hookrightarrow M_{2^\infty}, \tag{9}$$

we use this embedding and analysis of the cluster algebra $K_0(\mathbb{A}(\mathscr{D}))$ to prove the following result.

Theorem

The admissible values of index $[\mathscr{M} : \mathscr{N}]$ belong to the set

$$[\mathscr{M} : \mathscr{N}] \ \in \ [4, \infty) \bigcup \{4\cos^2\left(\frac{\pi}{n}\right) \mid n \geq 3\}. \tag{10}$$

.

[4] [Davidson 1996] [Example III.5.5]

Proof

To find admissible values of t, we shall use a simple analysis of the cluster algebra $K_0(\mathbb{A}(\mathscr{D}))$ based on the Sherman-Zelevinsky Theorem . Namely, recall that such an algebra has a canonical basis of the form

$$\mathcal{B} = \{x_i^p x_{i+1}^q \mid p, q \geq 0\} \bigcup \{T_n(x_1 x_4 - x_2 x_3) \mid n \geq 1\}, \qquad (11)$$

where $T_n(x)$ are the Chebyshev polynomials of the first kind.

The *Chebyshev polynomials of the first kind* (T_n) are given by

$$T_n(\cos(\theta)) = \cos(n\,\theta)\,.$$

For example, for $n = 2$ the T_2 formula can be converted into a polynomial with argument $x = \cos(\theta)$, using the double angle formula:

$$\cos(2\theta) = 2\cos^2(\theta) - 1$$

Replacing the terms in the formula with the definitions above, we get

$$T_2(x) = 2\,x^2 - 1\,.$$

For $x \neq 0$,

$$T_n\left(\frac{x + x^{-1}}{2}\right) = \frac{x^n + x^{-n}}{2}$$

Lemma 1

Lemma

The value $[\mathscr{M} : \mathscr{N}] \in (4, \infty)$ *is admissible.*

Proof.

Indeed, from (7) we have $t > 1$. But $[\mathscr{M} : \mathscr{N}] = \frac{(1+t)^2}{t}$, see formula (2). Therefore, one gets $[\mathscr{M} : \mathscr{N}] > 4$.

□

Lemma 2

Lemma

The value $t \in \{4\cos^2\left(\frac{\pi}{n}\right) \mid n \geq 3\} \cup \{4\}$ *is admissible.*

Proof of Lemma 2

Recall that

$$T_0 = 1 \quad \text{and} \quad T_n\left[\frac{1}{2}(t+t^{-1})\right] = \frac{1}{2}(t^n+t^{-n}). \qquad (12)$$

Proof of Lemma 2

Thus we shall look for a t such that $\frac{1}{2}(t + t^{-1}) = x_1 x_4 - x_2 x_3$. This is always possible since the Penner coordinates [Williams 2014] [8, Section 3.2] on $T_{\mathscr{D}}$ are given by the cluster (x_1, x_2), where each x_i is a function of t.

Exchange relations (4) for $\mathcal{A}(\mathscr{D})$ can be written as

$$x_{i-1} x_{i+1} = x_i^2 + 1$$

.

and we can calculate the following:

$$x_1 x_4 - x_2 x_3 = \frac{x_1^2 + 1 + x_2^2}{x_1 x_2}$$

.

From $x_1 x_4 - x_2 x_3 = \frac{1}{2}(t + t^{-1})$ one gets a system of equations:

$$\begin{cases} x_1 x_2 & = & 2t \\ x_1^2 + x_2^2 & = & t^2. \end{cases} \tag{14}$$

Excluding $x_2 = \frac{2t}{x_1}$ in the second equation, we get an equation:

$$x_1^4 - t^2 x_1^2 + 4t^2 = 0. \tag{15}$$

An explicit resolution of cluster variables x_1 and x_2 is given by the formulas:

$$\begin{cases} x_1 &= \frac{\sqrt{2}}{2}\sqrt{t^2 + t\sqrt{t^2 - 16}} \\ x_2 &= \frac{\sqrt{2}}{2}\sqrt{t^2 - t\sqrt{t^2 - 16}} \end{cases} \tag{16}$$

and the required equality $x_1 x_4 - x_2 x_3 = \frac{1}{2}(t + t^{-1})$ holds true in this case.

Proof of Lemma 2

The discrete values of t correspond to a finite cluster algebra $\mathcal{A}(\mathscr{D})$. Since the number of generators x_i is finite, we have $|\mathcal{B}| < \infty$. In particular, from the second series in (20) one obtains

$$T_n(x_1x_4 - x_2x_3) = T_0 = 1 \tag{17}$$

for some integer $n \geq 1$. But $x_1x_4 - x_2x_3 = \frac{1}{2}(t + t^{-1})$ and using formula (12) for the Chebyshev polynomials, one gets an equation

$$t^n + t^{-n} = 2 \tag{18}$$

for (possibly complex) values of t. Since (18) is equivalent to the equation $t^{2n} - 2t^n + 1 = (t^n - 1)^2 = 0$, one gets the n-th root of unity

$$t \in \{e^{\frac{2\pi i}{n}} \mid n \geq 1\}. \tag{19}$$

Proof of Lemma 2

The value

$$[\mathscr{M} : \mathscr{N}] = \frac{(1+t)^2}{t} = \frac{1}{t}+2+t = 2\left[\cos\left(\frac{2\pi}{n}\right)+1\right] = 4\cos^2\left(\frac{\pi}{n}\right) \quad (20)$$

is a real number. We must exclude the case $n = 2$ corresponding to the value $t = -1$, because otherwise one gets a division by zero in (1).
Sherman-Zelevinsky Theorem implies that there are no other admissible values of $[\mathscr{M} : \mathscr{N}]$. Therefore, our theorem follows from lemmas 1 and 2.

arXiv:1801.05510

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